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FINITELY GENERATED NILPOTENT GROUPS WITH ISOMORPHIC FINITE QUOTIENTS

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Abstract. Let $\mathscr{F}(G)$ denote the set of isomorphism classes of finite homomorphic images of a group G. We say that groups G and H have isomorphic finite quotients if $\mathscr{F}(G)=\mathscr{F}(H)$. In this paper we show that if G is a finitely generated nilpotent group, the finitely generated nilpotent groups H for which $\mathscr{F}(G)=\mathscr{F}(H)$ lie in only finitely many isomorphism classes. This is done using some finiteness results from the theory of algebraic groups along with some heretofore unpublished results of A. Borel.

Although the structure of finitely generated nilpotent groups is rather simple, the isomorphism problem for these groups is as yet unresolved. One method which has been considered for attacking the problem is the use of the set $\mathscr{F}(G)$ of isomorphism classes of finite quotients of the group G. The study of the finite quotients of finitely generated nilpotent groups has given much information about the structure of the groups [3], [8], although the finite quotients do not determine the group up to isomorphism ([15], and Higman, unpublished). The following positive result in this direction was obtained recently by A. Borel in some unpublished work: If G is a finitely generated torsion-free nilpotent group, then the finitely generated torsion-free nilpotent groups H for which $\mathscr{F}(G)=\mathscr{F}(H)$ are contained in finitely many commensurability classes. (Recall that two groups G and H are said to be commensurable if there is a group K which is isomorphic to a subgroup of finite index in each of G and H.)

The major part of this paper is devoted to showing that in a given commensurability class there can be only finitely many isomorphism classes of torsion-free finitely generated nilpotent groups with the same set of finite quotients. Borel's result and a relatively easy reduction to the torsion-free case then yield the following result which was announced in [14].

THEOREM. *Let G be a finitely generated nilpotent group. Then the finitely generated nilpotent groups H for which $\mathscr{F}(G)=\mathscr{F}(H)$ are contained in finitely many isomorphism classes.*

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This result implies that given a presentation of a finitely generated nilpotent group G, there is a finite list of presentations including the given one which contains exactly one representative for each isomorphism class of groups having the same finite quotients as G. If one could construct such a list algorithmically, one would have a solution to the isomorphism problem. The techniques of this paper do not yield such an effective procedure although it may be possible to produce a finite effective list containing at least one representative of each isomorphism class having the same finite quotients by our methods.

I would like to thank A. Borel for his permission to reproduce his heretofore unpublished results (Lemma 1.2 and Theorem 3.1) and G. Baumslag for many helpful conversations.

1. **Preliminaries on completions.** Let G be a finitely generated nilpotent group. Consider the infinite sequences $\{a_i\}$ of elements of G, for which

$$a_i^{-1}a_{i+1} \in G^{p^i} = \operatorname{gp}\{x^{p^i} \mid x \in G\}.$$

We will say two such sequences $\{a_i\}$ and $\{b_i\}$ are equivalent if $a_i^{-1}b_i$ is in G^{p^i} for each $i>0$. Note that the equivalence class of a sequence is unchanged if a finite number of terms are deleted (and the remaining terms renumbered in the same order). The set of equivalence classes of these sequences forms a nilpotent group under coordinatewise multiplication. We will call this group the p-adic completion of G and denote it by Z_pG. Z_pG is the completion of G with respect to the uniform topology, for which a neighborhood basis of the identity is given by the groups G^{p^i}. There is an integer k such that every element of G^{p^i} is a p^{i-k} power in G if $i>k$ [3]. Thus we could have used the sets $\{G\}^{p^i}=\{x^{p^i} \mid x \in G\}$ as a neighborhood basis of the identity for the p-adic topology. If G is residually a finite p-group, in particular if G is torsion free [6], the p-adic topology is Hausdorff and G is canonically included in Z_pG. If G is torsion free, then Z_pG must be torsion free.

Suppose now that H is a subgroup of G. Since H is of finite index in its p-isolator $\bar{H}$ [10, p. 248], there is an integer l such that $\bar{H}^{p^{i+l}} \subset H^{p^i}$ for each i. Using the integer k introduced above, we have

$$G^{p^{i+k+l}} \cap H \subset G^{p^{i+k+l}} \cap \bar{H} \subset \bar{H}^{p^{i+l}} \subset H^{p^i}.$$

Let $\{a_i\}$ be a sequence of the type introduced above, for which each a_i is in H. Then the sequence $\{n_i=a_{i+k+l}\}$ satisfies $n_i^{-1}n_{i+1} \in H^{p^i}$ for all i and is equivalent in G to the original sequence $\{a_i\}$. Thus Z_pH can be identified with the subgroup of Z_pG consisting of those equivalence classes containing a sequence with all its terms in H. If H is normal in G, Z_pH is normal in Z_pG. The following lemma then follows by appropriate modification of sequences:

Lemma 1.1. *Suppose G is a finitely generated nilpotent group. If H is a normal subgroup of G, then Z_pH is a normal subgroup of Z_pG and $Z_p(G/H)$ is isomorphic to Z_pG/Z_pH.*

The relation of p-adic completions to the problem with which we are concerned is given by the following lemma of Borel.

LEMMA 1.2. *Let G and H be finitely generated nilpotent groups. Then $\mathscr{F}(G)=\mathscr{F}(H)$ if and only if Z_pG is isomorphic to Z_pH for all finite primes p.*

Proof. For any positive integer m, we let $G^m=\text{gp}\{x^m \mid x\in G\}$, the smallest normal subgroup of G containing the elements x^m for all x in G. If Γ_m is defined to be G/G^m, Γ_m is a finite group of exponent m [10, p. 230] and every quotient of G of exponent dividing m is a quotient of Γ_m. In particular, if m divides n, we have $G^m\supset G^n$, so there is a canonical epimorphism $\gamma_{n,m}\colon \Gamma_n\to\Gamma_m$. Similarly we have H^m, Θ_m, and $\theta_{n,m}$. Since each finite quotient of G of exponent m is a quotient of Γ_m and similarly for H and Θ_m, G and H have isomorphic finite quotients if and only if Γ_m is isomorphic to Θ_m for each integer m. Every finite nilpotent group is the direct product of Sylow subgroups [10, p. 216], so Γ_m is isomorphic to Θ_m if and only if their respective Sylow subgroups are isomorphic. If p^i exactly divides m, then the p-Sylow subgroup of Γ_m is of exponent p^i and must be the largest quotient of Γ_m of exponent p^i (since Γ_m is a product of its Sylow subgroups). Thus the p-Sylow subgroup of Γ_m must be Γ_{p^i}. We then have that $\mathscr{F}(G)=\mathscr{F}(H)$ if and only if Γ_m is isomorphic to Θ_m for m a prime power.

We now restrict our attention to a particular prime p, and denote Γ_{p^i} by $\Gamma_{(i)}$ and similarly for Θ_{p^i} and the homomorphisms $\gamma_{n,m}$ and $\theta_{n,m}$. Suppose now that $\Gamma_{(i)}$ is isomorphic to $\Theta_{(i)}$ for each i. We claim that there must exist isomorphisms $f_j\colon \Gamma_{(j)}\to\Theta_{(j)}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\Gamma_{(1)} \arrow[d, "f_1"] & \Gamma_{(2)} \arrow[l, "\gamma_{(2),(1)}"'] \arrow[d, "f_2"] & \cdots \arrow[l] & \Gamma_{(j)} \arrow[l] \arrow[d, "f_j"] & \Gamma_{(j+1)} \arrow[l, "\gamma_{(j+1),(j)}"'] \arrow[d, "f_{j+1}"] & \cdots \arrow[l] \\
\Theta_{(1)} & \Theta_{(2)} \arrow[l, "\theta_{(2),(1)}"'] & \cdots \arrow[l] & \Theta_{(j)} \arrow[l] & \Theta_{(j+1)} \arrow[l, "\theta_{(j+1),(j)}"'] & \cdots \arrow[l]
\end{tikzcd}$$

Let f_i be an isomorphism, $f_i\colon \Gamma_{(i)}\to\Theta_{(i)}$. We will say that f_i extends to $j>i$ if there is an isomorphism f_j such that the following diagram commutes:

$$\begin{tikzcd}
\Gamma_{(i)} \arrow[d, "f_i"] & \Gamma_{(j)} \arrow[l, "\gamma_{(j),(i)}"'] \arrow[d, "f_j"] \\
\Theta_{(i)} & \Theta_{(j)} \arrow[l, "\theta_{(j),(i)}"']
\end{tikzcd}$$

and that f_i is indefinitely extendable if f_i extends to j for each $j>i$.

For $i\leqq j$, any isomorphism $f_j\colon \Gamma_{(j)}\to\Theta_{(j)}$ is an extension of some f_i since $\Gamma_{(i)}$ and $\Theta_{(i)}$ are the largest quotients of $\Gamma_{(j)}$ and $\Theta_{(j)}$, respectively, of exponent p^i. Thus for a given i and any $j\geqq i$, some isomorphism $f_i\colon \Gamma_{(i)}\to\Theta_{(i)}$ extends to j. Since the set of isomorphisms of $\Gamma_{(i)}$ with $\Theta_{(i)}$ is finite, some such f_i must extend to infinitely many and thus to all $j>i$. We thus have that $\mathscr{F}(G)=\mathscr{F}(H)$ implies that $Z_pG=\text{proj lim}\,(\Gamma_{(i)}, \gamma_{(j),(i)})$ is isomorphic to $Z_pH=\text{proj lim}\,(\Theta_{(i)}, \theta_{(j),(i)})$.

Conversely if Z_pG is isomorphic to Z_pH we have

$$\Gamma_{(i)} = G/G^{p^i} \cong Z_pG/(Z_pG)^{p^i} \cong Z_pH/(Z_pH)^{p^i} \cong H/H^{p^i} = \Theta_{(i)},$$

which completes the proof.

The remainder of this section details properties of finitely generated torsion-free nilpotent groups, which we will term N-groups. Recall that every torsion-free nilpotent group G may be embedded in a divisible nilpotent group G^* of the same class [10, p. 256]. For such an embedding, the intersection of all divisible subgroups of G^* containing G is called a Mal'cev completion of G. The Mal'cev completion of G is unique up to isomorphism and has the property that each of its elements has some power lying in G [10, p. 256]. For a given N-group G, we will denote its Mal'cev completion by QG and the Mal'cev completion of Z_pG by Q_pG. We then have G canonically included in each of Z_pG and QG and each of Z_pG and QG canonically included in Q_pG. Two N-groups G and H are commensurable if and only if QG is isomorphic to QH [12].

For any λ in Z_p, the ring of p-adic integers, there are rational integers $\{a(i)\}_{i=0}^{\infty}$ such that $a(i)-a(i+1)$ is divisible by p^i and the sequence $\{a(i)\}$ converges to λ in Z_p. For any x in Z_pG, the sequence $\{x^{a(i)}\}$ is a Cauchy sequence. Since Z_pG is complete, this sequence has a limit which we will call x^λ. This operation makes Z_pG into a Z_p-group; that is, for each λ in Z_p and x in Z_pG, there is a unique element x^λ in Z_pG, defined in such a way that $x^\lambda x^\mu=x^{\lambda+\mu}$, $(x^\lambda)^\mu=x^{\lambda\mu}$ and $(y^{-1}xy)^\lambda=y^{-1}x^\lambda y$ for all λ, μ in Z_p and x, y in Z_pG (see [11]). Note that if A is any abelian, Z_p-closed subgroup of Z_pG, e.g., if A is the center of Z_pG, the above operation makes A into a Z_p-module. It follows from the properties of Mal'cev completions that QG is a Q-group, where Q denotes the rational numbers, and that Q_pG is a Q_p-group, where Q_p denotes the p-adic rational numbers.

Recall that a normal basis for an N-group G is an ordered finite set $(x_1, \ldots, x_m)$ of elements of G such that

(a) each element of G may be written uniquely in the form $x_1^{n(1)}x_2^{n(2)}\cdots x_m^{n(m)}$ for integers $n(i)$,

(b) each x_i is central modulo $\text{gp}\,\{x_{i+1}, \ldots, x_m\}$.

Such normal bases exist for any N-group [7]. The following lemma, which is essentially known [7], shows that $(x_1, \ldots, x_m)$ is an X-normal basis for XG, where X is Z_p, Q or Q_p. The proof is an easy induction on the torsion-free rank m of G, using Lemma 1.1, and the fact that if x^m and y^n commute in a torsion-free nilpotent group, then x and y also commute [10, p. 243].

LEMMA 1.3. *Let $(x_1, \ldots, x_m)$ be a normal basis for an N-group G. Then*

(a) *every element of Z_pG may be written uniquely in the form $x_1^{r(1)}\cdots x_m^{r(m)}$ with $r(i)$ in Z_p,*

(b) *every element of QG may be written uniquely in the form $x_1^{r(1)} \cdots x_m^{r(m)}$ with $r(i)$ in Q,*

(c) *every element of Q_pG may be written uniquely in the form $x_1^{r(1)} \cdots x_m^{r(m)}$ with $r(i)$ in Q_p.*

Suppose x is an element of QG. We will say that x lies in Z_pG if, whenever x is considered as an element of Q_pG, via the canonical inclusion, x is an element of Z_pG. We then have

COROLLARY 1.4. *The elements of QG which lie in Z_pG for all primes p are precisely the elements of G.*

Proof. Let x be such an element. Then x can be written uniquely as $x_1^{r(1)} \cdots x_m^{r(m)}$ with $r(i)$ in Q and $r(i)$ in Z_p for each prime p. This implies that $r(i)$ is rational integer for each i, so that x is in G.

LEMMA 1.5. *Any homomorphism $\varphi: Z_pG \to Z_pH$ is a Z_p-homomorphism; that is $\varphi(x^\lambda) = \varphi(x)^\lambda$ for λ in Z_p and x in Z_pG.*

Proof. The groups $(Z_pG)^{p^i}$ and $(Z_pH)^{p^i}$ form neighborhood bases for the identity in Z_pG and Z_pH respectively. Since $\varphi((Z_pG)^{p^i}) \subset (Z_pH)^{p^i}$, φ is continuous in the respective topologies. This means that φ preserves limits so that the lemma follows from the definition of x^λ.

LEMMA 1.6. *Let G and H be N-groups. Then any isomorphism φ of G onto H extends uniquely to isomorphisms φ_p of Z_pG onto Z_pH and to an isomorphism $\bar{\varphi}$ of QG onto QH. Further, an isomorphism φ_p of Z_pG onto Z_pH extends uniquely to an isomorphism $\bar{\varphi}_p$ of Q_pG onto Q_pH, and an isomorphism ψ of QG onto QH extends uniquely to isomorphisms ψ_p of Q_pG onto Q_pH.*

Proof. This follows from Theorem 6.7 of [7] and Lemma 1.5.

COROLLARY 1.7. *Let G and H be N-groups for which there exist isomorphisms $\varphi_p: Z_pG \to Z_pH$ and $\psi: QG \to QH$ which are onto and such that the extensions $\bar{\varphi}_p$ and ψ_p to Q_pG are the same for all primes p. Then G and H are isomorphic by a map φ which extends to φ_p and ψ on the respective groups.*

Proof. If x is in G, we have that $\psi(x)$ is an element of QH which is in Z_pH for all p, and thus is in H so that $\psi(G) \subset H$. Similarly $\psi^{-1}(H) \subset G$, so that ψ restricted to G must be an isomorphism onto H.

LEMMA 1.8. *Let G be an N-group and let H be a subgroup of finite index in G. Then Z_pH is of finite index in Z_pG and, for all but a finite number of primes p, $Z_pH = Z_pG$.*

Proof. Suppose the index of H in G is n and that $(x_1, \ldots, x_m)$ is a normal basis for G. Then for each i, there is an integer $k(i) \leq n$ such that $x_i^{k(i)}$ is in H. If p is a prime greater than n, $k(i)$ is invertible in Z_p so there is an element $\lambda(i)$ of Z_p such that $k(i)\lambda(i)=1$. This implies that $(x_i^{k(i)})^{\lambda(i)}=x_i$ is an element of Z_pH, for each i. It follows from Lemma 1.3 that $Z_pH=Z_pG$ for primes p greater than n.

Since every subgroup of a nilpotent group G is subnormal, it follows that if H is a subgroup of index n in G and x is an arbitrary element of G, then x^n is in H. If λ is an element of Z_p, then $\lambda=j+n\lambda'$ with $0\leqq j<n$ and λ' in Z_p. We will show by induction on the torsion-free rank m that each element $x_1^{\lambda(1)}\cdots x_m^{\lambda(m)}$ in Z_pG has a coset representative with respect to Z_pH of the form $x_1^{j(1)}\cdots x_m^{j(m)}$ with integers $j(i)<n$. The case $m=1$ is obvious and by the inductive hypothesis applied to $G/\mathrm{gp}\,\{x_m\}$, we have

$$x_1^{\lambda(1)}\cdots x_{m-1}^{\lambda(m-1)}x_m^{\lambda(m)} = x_1^{j(1)}\cdots x_{m-1}^{j(m-1)}x_m^{\mu(m)}h,$$

for some $\mu(m)$ in Z_p and some h in Z_pH. As we noted above $\mu(m)=j(m)+n\lambda'$ so that $x_m^{\mu(m)}=x_m^{j(m)}(x_m^n)^{\lambda'}$. This completes the inductive step, implying that each coset of Z_pH has a representative in G. Thus the number of cosets of Z_pH in Z_pG can be no greater than the number n of cosets of H in G.

LEMMA 1.9. *Let G and H be N-groups and let φ be an isomorphism of QG onto QH. Then for all but a finite number of primes p, the extension of φ to an isomorphism φ_p of Q_pG onto Q_pH sends Z_pG isomorphically onto Z_pH.*

Proof. $\varphi(G)$ is a subgroup of QH commensurable with H, so there is a subgroup K of QH, which is of finite index in each of H and $\varphi(G)$. By Lemma 1.8, for all but a finite number of primes p, $Z_p(\varphi(G))=Z_pK$ and $Z_pH=Z_pK$ so that $Z_p(\varphi(G))=Z_pH$. Since $Z_p(\varphi(G))=\varphi_p(Z_pG)$, the lemma follows.

LEMMA 1.10. *Let G be an N-group. Then there are only finitely many subgroups of Q_pG which contain Z_pG as a subgroup of a given finite index n.*

Proof. Let $G^*=\mathrm{gp}\,\{x\in Q_pG \mid x^n\in Z_pG\}$. If H is any subgroup of Q_pG containing Z_pG as a subgroup of index n, H is contained in G^*. We will show by induction on the torsion-free rank m of G that Z_pG is of finite index in G^* so there can be only a finite number of groups between Z_pG and G^*. Let $(x_1,\ldots,x_m)$ be a normal basis for G. Let W be the Q_p-subgroup of Q_pG generated by x_m and let $Z^*=G^*\cap W$ and $Z=Z_pG\cap W$. As noted before Lemma 1.3, each of Z, Z^* and W is a Z_p-module and Z is singly generated (by x_m). For any x in G^* we have x^{n^c} in Z_pG [10, p. 248] so $(Z^*)^{n^c}\subset Z$. Thus Z^* is a singly generated Z_p-module and Z is of finite index in Z^* (by properties of Z_p-modules). The inductive hypothesis implies that $Z^*\cdot(Z_pG)$ is of finite index in G^*, so that Z_pG is of finite index in G^* as claimed.

LEMMA 1.11. *Let G be an N-group. Then there are only finitely many subgroups of Z_pG of a given finite index n.*

Proof. For any subgroup H of index n in Z_pG, H must contain $(Z_pG)^n$. By the same method as in Lemma 1.10 we may show that $(Z_pG)^n$ is of finite index in Z_pG and the lemma follows as before.

If H is a subgroup of a group G and X is a group of automorphisms of G, we will denote the group of automorphisms in X which take H isomorphically onto itself by stab (H, X).

LEMMA 1.12. *Let G be an N-group and let H be a subgroup of Q_pG, which contains Z_pG as a subgroup of finite index. Then* stab $(H, \text{stab}\,(Z_pG, \text{Aut}\,(Q_pG)))$ *is of finite index in* stab $(Z_pG, \text{Aut}\,(Q_pG))$.

Proof. Let n be the index of Z_pG in H and let $H=H_1, H_2, \ldots, H_k$ be a list (finite by Lemma 1.10) of the subgroups of Q_pG containing Z_pG as a subgroup of index n. If φ is any element of stab $(Z_pG, \text{Aut}\,(Q_pG))$, φ permutes the groups H_i. Thus we may construct a homomorphism from stab $(Z_pG, \text{Aut}\,(Q_pG))$ into a finite permutation group. The kernel of this homomorphism is of finite index in stab $(Z_pG, \text{Aut}\,(Q_pG))$ and is contained in stab $(H, \text{stab}\,(Z_pG, \text{Aut}\,(Q_pG)))$.

By an analogous argument, we may prove

LEMMA 1.13. *Let G be an N-group and let H be a subgroup of finite index in Z_pG. Then* stab $(H, \text{stab}\,(Z_pG, \text{Aut}\,(Q_pG)))$ *is of finite index in* stab $(Z_pG, \text{Aut}\,(Q_pG))$.

PROPOSITION 1.14. *Let H be a subgroup of finite index in an N-group G. Then the subgroups* stab $(Z_pG, \text{Aut}\,(Q_pG))$ *and* stab $(Z_pH, \text{Aut}\,(Q_pG))$ *of* Aut (Q_pG) *intersect in a subgroup K of finite index in each.*

Proof.

$$K = \text{stab}\,(Z_pG, \text{stab}\,(Z_pH, \text{Aut}\,(Q_pG))) = \text{stab}\,(Z_pH, \text{stab}\,(Z_pG, \text{Aut}\,(Q_pG))).$$

By Lemma 1.8, Z_pH is of finite index in Z_pG so the proposition follows immediately from Lemmas 1.12 and 1.13.

2. **Lie algebras and algebraic groups.** In this section we recall facts about N-groups and their related Lie algebras [9], [12]. We then extend these results to the p-adic completions to obtain a relation between various automorphism groups and an algebraic matric group.

Let $\mathscr{R}$ be the group ring of an N-group G over the rational numbers and let $\mathscr{A}$ be the augmentation ideal of $\mathscr{R}$. Then $\mathscr{A}$ is residually nilpotent; that is $\bigcap_{n=1}^{\infty} \mathscr{A}^n=0$ [9, Theorem 4.3]. We may thus define a Hausdorff $\mathscr{A}$-adic topology on $\mathscr{R}$ for which a neighborhood basis of zero is given by the powers $\{\mathscr{A}^n\}$ of $\mathscr{A}$. We will denote the completion of $\mathscr{R}$ in this topology by $\mathscr{R}^\wedge$ and the completion of $\mathscr{A}$ by $\mathscr{A}^\wedge \subset \mathscr{R}^\wedge$. We also have $\mathscr{R} \subset \mathscr{R}^\wedge$. For any element x in $\mathscr{A}^\wedge$, we may form the Cauchy series

$$1+x+x^2/2!+x^3/3!+\cdots$$

which converges in $\mathscr{R}^\wedge$ to a limit which we will call exp (x), an element of $1+\mathscr{A}^\wedge$. Similarly if y is in $\mathscr{A}^\wedge$ we have the series

$$y-1/2y^2+1/3y^3-\cdots$$

whose limit we will denote by $\log(1+y)$. We have as usual that $\log(\exp(x))=x$ and $\exp(\log(1+y))=1+y$. Since G is contained in the set $1+\mathscr{A}^\wedge$, we may consider the Q-subspace Λ of $\mathscr{A}^\wedge$ spanned by $\log(G)$. The dimension of Λ is the torsion-free rank of G and Λ is a nilpotent Lie subalgebra of the commutation Lie algebra in $\mathscr{R}^\wedge$ [9]. We call Λ the Lie algebra of G.

Recall that by using the Baker-Campbell-Hausdorff formula, we may define a multiplication $*$ in any nilpotent Lie algebra Λ over a field F of characteristic zero by

$$x * y = x+y+(1/2)(x, y)+(1/12)((x, y), y)+(1/12)((y, x), x)+\cdots$$

where (,) denotes the Lie product in Λ and the sum is finite since Λ is nilpotent. The multiplication $*$ makes Λ into a nilpotent F-group (in the sense of §1, via scalar multiplication) [9], which we will denote by $(\Lambda, *)$. Let $[a, b]$ denote the group theoretic commutator of a and b in $(\Lambda, *)$ and (a, b) denote the Lie product of a and b in Λ, with similar notation for higher order commutators or Lie products. Then we have for any commutator [9]:

(α) $[g_1, \ldots, g_k]=(g_1, \ldots, g_k)+$ higher order Lie products in which each g_i occurs at least once.

For the Lie algebra Λ of an N-group G, we have for a and b in $\log G$, $\exp(a * b) = \exp(a)\cdot\exp(b)$. Thus exp gives an isomorphism of the set $\log G$ with multiplication $*$ onto G. In fact exp gives an isomorphism of $(\Lambda, *)$ with a Mal'cev completion QG of G. Also automorphisms of G are in one-to-one correspondence with automorphisms of Λ which stabilize $\log G$ [2, Chapter 4].

LEMMA 2.1. *Let Λ and Γ be nilpotent Lie algebras over a field F of characteristic zero, and suppose $\varphi\colon \Lambda \to \Gamma$ is a one-to-one and onto map of sets. Then φ is an isomorphism of Λ onto Γ as Lie algebras if and only if it is an isomorphism of $(\Lambda, *)$ onto $(\Gamma, *)$ as F-groups.*

Proof. The "only if" part follows directly from the fact that the $*$ multiplication is defined in terms of sums and Lie products and the fact that the action of F is given by scalar multiplication, since sums, Lie products, and scalar multiplication must be preserved by a Lie algebra homomorphism.

Conversely, if φ preserves the action of F, then φ preserves scalar multiplication. To finish the proof we must show that $\varphi(a+b)=\varphi(a)+\varphi(b)$ and $\varphi((a, b))=(\varphi(a),\varphi(b))$ for all a and b in Λ. We do this for a in Λ and b in $\mathscr{Z}_i(\Lambda)$, the ith center of the Lie algebra Λ, by induction on i. It follows easily from (α) above that b is in $\mathscr{Z}_i(\Lambda)$ if and only if b is in $Z_i(\Lambda, *)$, the ith center of the group $(\Lambda, *)$. Thus since φ is a group homomorphism, if b is in $\mathscr{Z}_i(\Lambda)$, $\varphi(b)$ is in $\mathscr{Z}_i(\Gamma)$.

For b in $\mathscr{Z}_1(\Lambda)$, all Lie products in Λ in which b occurs are zero, and all Lie products in Γ in which $\varphi(b)$ occurs are zero. Thus we have

$$\begin{aligned} a * b &= a+b, \\ \varphi(a+b) &= \varphi(a * b) = \varphi(a) * \varphi(b) = \varphi(a)+\varphi(b), \\ \varphi(a, b) &= \varphi(0) = 0 = (\varphi(a), \varphi(b)), \end{aligned}$$

so we may begin the induction.

Now we assume that $\varphi(a+b)=\varphi(a)+\varphi(b)$ and $\varphi(a, b)=(\varphi(a), \varphi(b))$ for any a in Λ and any b in $\mathscr{Z}_i(\Lambda)$ for which i is less than n, and prove the same statement for b in $\mathscr{Z}_n(\Lambda)$. We then have $a * b=a+b+(1/2)(a, b)+\cdots+n$-fold Lie products in a and b. Any Lie product containing b must lie in $\mathscr{Z}_i$ for $i<n$, so we may apply the induction hypothesis to obtain

$$\varphi(a * b) = \varphi(a+b+(1/2)(a, b)+\cdots) = \varphi(a+b)+(1/2)\varphi(a, b)+\cdots.$$

We may show that φ preserves Lie products involving j terms, one of which is b, by a downward induction starting with $j=n$. The proof uses the fact that φ preserves group theoretic commutators, our induction hypothesis and the statement (α) above. Using this fact we have

$$\varphi(a * b) = \varphi(a+b)+(1/2)(\varphi(a), \varphi(b))+\cdots.$$

We also have, however, that

$$\varphi(a * b) = \varphi(a) * \varphi(b) = \varphi(a)+\varphi(b)+(1/2)(\varphi(a), \varphi(b))+\cdots.$$

By canceling the Lie products, we obtain $\varphi(a+b)=\varphi(a)+\varphi(b)$, as desired.

Recall that an N-group G is said to be lattice nilpotent if $\log G$ is closed under addition (thus a lattice) in the Lie algebra Λ of G. For the remainder of this section (except Corollary 2.3) we will assume that G is lattice nilpotent. Suppose that we put a p-adic topology on the abelian group $\log G$, for which the neighborhood basis of zero is given by the subgroups $\{p^i \log G\}$ for integers $i>0$. If g is an element of G, we have $\exp(p^i \log g)=g^{p^i}$. Thus $\exp(p^i \log G)=\{G\}^{p^i}$, the set of p^i-powers in G. As noted in §1, the topology in G for which a neighborhood basis of the identity is given by the sets $\{G\}^{p^i}$ is in fact the p-adic topology in G. Thus exp is a homeomorphism from $\log G$ with its p-adic topology to G with its p-adic topology.

Let Λ_p denote the tensor product $Q_p \otimes_Q \Lambda = Q_p \otimes_Z \log G$ and denote by $Z_p \log G$ the abelian subgroup $Z_p \otimes_Z \log G$ of Λ_p. We may consider $\log G$ contained in $Z_p \log G$ as $1 \otimes \log G$ and Λ contained in Λ_p as $1 \otimes \Lambda$. We then have

PROPOSITION 2.2. *Let G be a lattice nilpotent group. Then* $\mathrm{Aut}(Q_pG)$ *is isomorphic to* $\mathrm{Aut}(\Lambda_p)$ *by an isomorphism which takes*
$\mathrm{stab}(QG, \mathrm{Aut}(Q_pG))$ *to* $\mathrm{stab}(\Lambda, \mathrm{Aut}(\Lambda_p))$,
$\mathrm{stab}(Z_pG, \mathrm{Aut}(Q_pG))$ *to* $\mathrm{stab}(Z_p \log G, \mathrm{Aut}(\Lambda_p))$, *and*
$\mathrm{stab}(G, \mathrm{Aut}(Q_pG))$ *to* $\mathrm{stab}(\log G, \mathrm{Aut}(\Lambda_p))$.

Proof. $Z_p \log G$ is clearly the completion of $\log G$ in its p-adic topology. Since Λ_p is a nilpotent Lie algebra, we may define the $*$ multiplication on it. Since exp is a homeomorphism and an isomorphism of $(\log G, *)$ onto G, it must extend to an isomorphism of $(Z_p \log G, *)$ onto Z_pG. $(\Lambda_p, *)$ is a divisible nilpotent group and some integral multiple of each element of Λ_p lies in $Z_p \log G$, so $(\Lambda_p, *)$ must be a Mal'cev completion of $(Z_p \log G, *)$. We thus have that $(\Lambda_p, *)$ is isomorphic to Q_pG by a further extension of exp. We may now define the desired isomorphism by sending an element φ of Aut (Q_pG) to $\log \circ \varphi \circ \exp$ in Aut $(\Lambda_p, *)$ which equals Aut (Λ_p) by Lemma 2.1. This isomorphism clearly does as required.

COROLLARY 2.3. *Suppose G is an arbitrary N-group and Λ is its Lie algebra. Then QG is isomorphic to $(\Lambda, *)$ and Q_pG is isomorphic to $(\Lambda_p, *)$.*

Proof. There is a lattice nilpotent group H which contains G as a subgroup of finite index [13, Theorem 2]. We then have QG isomorphic to QH, so that if the Lie algebra of H is Γ, we have $(\Lambda, *)$ isomorphic to $(\Gamma, *)$ and thus by Lemma 2.1 Λ isomorphic to Γ. Also Z_pG is of finite index in Z_pH so Q_pG and Q_pH are isomorphic. Thus we have

$$Q_pG \cong Q_pH \cong (\Gamma_p, *) \cong (\Lambda_p, *).$$

Recall [4] now that an algebraic matric group $\mathfrak{H}$ of degree n over a field k is given by an ideal $\mathscr{I}$ of polynomials in $k[X_{11}, \ldots, X_{nn}]$ such that for some (and hence every) algebraic closure $\bar{k}$ of k, the set of elements $\mathfrak{H}_{\bar{k}}$ of elements of GL $(n, \bar{k})$ whose entries annihilate $\mathscr{I}$ is a group. If B is any subring of an overfield of k, we denote by $\mathfrak{H}_B$ the group of elements of GL (n, B) whose entries annihilate $\mathscr{I}$.

Suppose now that Λ is a finite-dimensional rational Lie algebra with vector space basis $\{a_i\}$ and structure constants with respect to this basis $\{\gamma_{ijk}\}$. A matrix (X_{ij}) in GL (n, Q) acting on the basis $\{a_i\}$ gives an automorphism of Λ if and only if the entries of the matrix satisfy the equations

$$\sum_r \sum_s X_{ir}X_{js}\gamma_{rst} = \sum_k \gamma_{ijk}X_{kt} \quad \text{for all } i, j, t.$$

Thus the automorphism group of Λ is isomorphic to the group of rational points $\mathfrak{G}_Q$ of an algebraic matric group $\mathfrak{G}$ over Q. Similarly, Aut $(Q_p \otimes_Q \Lambda)$ is isomorphic to $\mathfrak{G}_{Q_p}$.

Now let Λ be the Lie algebra of our lattice nilpotent group G and take as a basis for Λ, a Z-basis for the lattice $\log G$. Let $\mathfrak{G}$ be the algebraic matric group obtained as above with respect to this basis. In view of Proposition 2.2, we have

PROPOSITION 2.4. *If G is a lattice nilpotent group, then there is an algebraic matric group $\mathfrak{G}$ over Q such that* Aut (Q_pG) *is isomorphic to $\mathfrak{G}_{Q_p}$ for each p, by an isomorphism which takes*

stab $(Z_pG$, Aut $(Q_pG))$ *to* $\mathfrak{G}_{Z_p}$,

stab $(QG$, Aut $(Q_pG))$ *to* $\mathfrak{G}_Q$, *and*

stab $(G$, Aut $(Q_pG))$ *to* $\mathfrak{G}_Z$

in such a way that if φ is an automorphism of QG and φ_p the extension of φ to an element of stab $(QG$, Aut $(Q_pG))$, *the element of $\mathfrak{G}_Q$ determined by φ_p is independent of p.*

Suppose now that $\mathfrak{G}$ is an algebraic matric group over Q, and let V be the set of finite primes in Z. We then define $\mathfrak{G}_A$ to be the subgroup of $\prod_{p \in V} \mathfrak{G}_{Q_p}$ consisting of elements $(x_p)_{p \in V}$ such that, for all but a finite number of primes p, x_p is in $\mathfrak{G}_{Z_p}$. If x is in $\mathfrak{G}_Q$, we know that x is in $\mathfrak{G}_{Z_p}$ for all but a finite number of primes p. Thus

$\mathfrak{G}_Q$ may be diagonally embedded in $\mathfrak{G}_A$ by $x \to (x)_{p \in V}$. We further define $\mathfrak{G}_A^\infty$ to be the subgroup of $\mathfrak{G}_A$ consisting of $(x_p)_{p \in V}$ for which x_p is in $\mathfrak{G}_{Z_p}$ for all primes p. With these definitions we have

THEOREM A [4, THEOREM 5.1]. *Let $\mathfrak{G}$ be an algebraic matric group over Q. Then the number $c(\mathfrak{G})$ of distinct double cosets $\mathfrak{G}_A^\infty \cdot x \cdot \mathfrak{G}_Q$ $(x \in \mathfrak{G}_A)$ is finite.*

3. Proof of theorem.

THEOREM 3.1 (BOREL). *Let G be an N-group. Then the N-groups H for which $\mathscr{F}(G)=\mathscr{F}(H)$ are contained in finitely many commensurability classes.*

Proof. Let G and H be N-groups and let Γ and Λ be their respective Lie algebras. We know that G and H are commensurable if and only if Γ and Λ are rationally isomorphic; that is, commensurability classes of N-groups are in one-to-one correspondence with isomorphism classes of rational nilpotent Lie algebras. We also know by Lemma 1.2 that if $\mathscr{F}(G)=\mathscr{F}(H)$, Z_pG and Z_pH must be isomorphic for each prime p. This implies that Q_pG and Q_pH are isomorphic and thus by Corollary 2.3 that Γ_p and Λ_p are isomorphic for each prime p. Thus if Λ is the Lie algebra of an N-group H, for which $\mathscr{F}(G)=\mathscr{F}(H)$, Λ must be isomorphic to Γ over Q_p for each prime p. By Theorem 7.11 of [5], there can be only finitely many isomorphism classes of such rational Lie algebras, and thus only finitely many commensurability classes containing an N-group H with $\mathscr{F}(G)=\mathscr{F}(H)$.

REMARK. Borel has pointed out that by using the full force of Theorem 7.11 of [5], one could strengthen Theorem 3.1 to say: If P is a finite set of primes and G is an N-group then the N-groups H, for which Z_pG is isomorphic to Z_pH for all primes p not in P, lie in only finitely many commensurability classes. For our main theorem, however, we need all primes as shown by the groups

$$F_n = \{a, b, c \mid [a, b] = c^{p^n}, \text{ and } F_n \text{ is nilpotent of class 2}\}.$$

These groups are nonisomorphic (consider F_n/F_n'), but for all primes $q \neq p$, Z_qF_n is isomorphic to Z_qF_m for all integers m and n.

Suppose that G is an N-group. We define the group $\mathscr{G}_A$ to be the subgroup of $\prod_{p \in V} \operatorname{Aut}(Q_pG)$ consisting of elements $\prod_{p \in V} (\alpha_p)$, such that, for all but a finite number of primes p, α_p is in stab $(Z_pG, \operatorname{Aut}(Q_pG))$. We define the group $\mathscr{G}_A^\infty$ to be the subgroup of $\mathscr{G}_A$ consisting of all $\prod_{p \in V} (\alpha_p)$ for which α_p is in stab $(Z_pG, \operatorname{Aut}(Q_pG))$ for all primes p. If α is in Aut (QG), we may consider α to be an element of $\mathscr{G}_A$ as follows: for each prime p in V let α_p be the unique extension of α to an automorphism of Q_pG. By Lemma 1.9, α_p is an element of stab $(Z_pG, \operatorname{Aut}(Q_pG))$ for all but a finite number of primes p, so that $\prod_{p \in V} (\alpha_p)$ is an element of $\mathscr{G}_A$. We will denote Aut (QG), embedded in $\mathscr{G}_A$ in the above manner, by $\mathscr{G}_Q$.

Given two N-groups G and H we will say that H is in the genus of G if H is commensurable with G and if H and G have isomorphic finite quotients. The following proposition was inspired by Proposition 2.3 of [4].

PROPOSITION 3.2. *The isomorphism classes in the genus of G are in one-to-one correspondence with a subset of the set of double cosets $\mathscr{G}_A^\infty\backslash\mathscr{G}_A/\mathscr{G}_Q$.*

Proof. Suppose that H is in the genus of G. Then by Lemma 1.2, there are isomorphisms $\varphi_p\colon Z_pG\to Z_pH$ for each p in V as well as an isomorphism $\psi\colon QG\to QH$. Each φ_p extends to an isomorphism $\bar\varphi_p\colon Q_pG\to Q_pH$ and, for each prime p, ψ extends to an isomorphism $\psi_p\colon Q_pG\to Q_pH$. By Lemma 1.9, ψ_p maps Z_pG isomorphically onto Z_pH for all but a finite number of primes p, so that $\prod_{p\in V}(\bar\varphi_p^{-1}\circ\psi_p)$ is an element of $\mathscr{G}_A$. We now show that the corresponding double coset in $\mathscr{G}_A^\infty\backslash\mathscr{G}_A/\mathscr{G}_Q$ is independent of several choices made above. First suppose that θ_p were a different choice of isomorphism from Z_pG to Z_pH. Then $\theta_p=\varphi_p\circ\varphi_p^{-1}\circ\theta_p$ and $\theta_p^{-1}\circ\varphi_p$ is an automorphism of Z_pG for each prime p. Thus we have $\prod_{p\in V}(\bar\theta_p^{-1}\circ\bar\varphi_p)$ is an element of $\mathscr{G}_A^\infty$ so that $\prod_{p\in V}(\bar\theta_p^{-1}\circ\psi_p)$ lies in the same double coset as $\prod_{p\in V}(\bar\varphi_p^{-1}\circ\psi_p)$. Similarly if γ were a different choice of isomorphism from QG to QH, we would have $\gamma=\psi\circ\psi^{-1}\circ\gamma$. Again since $\psi^{-1}\circ\gamma$ is in Aut (QG), we have $\prod_{p\in V}(\bar\varphi_p^{-1}\circ\gamma_p)$ in the same double coset as $\prod_{p\in V}(\bar\varphi_p^{-1}\circ\psi_p)$. Finally, suppose K were a group isomorphic to H by an isomorphism σ. Then σ extends to isomorphisms $\sigma_p\colon Z_pH\to Z_pK$ and $\bar\sigma\colon QH\to QK$, each of which extends to the same isomorphism $\bar\sigma_p\colon Q_pH\to Q_pK$, for a given prime p. We then have $\sigma_p\circ\varphi_p\colon Z_pG\to Z_pK$ and $\bar\sigma\circ\psi\colon QG\to QK$. The element of $\mathscr{G}_A$ corresponding to these isomorphisms is $\prod_{p\in V}(\bar\varphi_p^{-1}\circ\bar\sigma_p^{-1}\circ\bar\sigma_p\circ\psi_p)=\prod_{p\in V}(\bar\varphi_p^{-1}\circ\psi_p)$. We thus have that the above defined map from isomorphism classes in the genus of G to the double cosets $\mathscr{G}_A^\infty\backslash\mathscr{G}_A/\mathscr{G}_Q$ is well defined.

Now suppose H and K are two N-groups in the genus of G, whose respective isomorphism classes are sent by the above defined map to the same double coset $\mathscr{G}_A^\infty\backslash\mathscr{G}_A/\mathscr{G}_Q$. We will show that H and K must be isomorphic, proving that our set map is one-to-one and thus giving the desired conclusion. Let $\varphi_p\colon Z_pG\to Z_pH$, $\psi\colon QG\to QH$, $\sigma_p\colon Z_pG\to Z_pK$ and $\tau\colon QG\to QK$ be isomorphisms. By hypothesis, $\prod_{p\in V}(\bar\varphi_p^{-1}\circ\psi_p)$ and $\prod_{p\in V}(\bar\sigma_p^{-1}\circ\tau_p)$ lie in the same double coset. Thus we may modify σ_p by an automorphism of Z_pG and ψ by an automorphism of QG so that $\prod_{p\in V}(\bar\varphi_p^{-1}\circ\psi_p)=\prod_{p\in V}(\bar\sigma_p^{-1}\circ\tau_p)$. This means that for each prime p, $\bar\varphi_p^{-1}\circ\psi_p=\bar\sigma_p^{-1}\circ\tau_p$ as automorphisms of Q_pG. Consider now the isomorphisms $\theta_p=\sigma_p\circ\varphi_p^{-1}\colon Z_pH\to Z_pK$ and $\gamma=\tau\circ\psi^{-1}\colon QH\to QK$. We then have

$$\bar\theta_p=\bar\sigma_p\circ\bar\varphi_p^{-1}=\tau_p\circ\psi_p^{-1}=\gamma_p\colon Q_pH\to Q_pK,$$

by the above remarks. By Corollary 1.7, the groups H and K must be isomorphic.

We now show that the number of double cosets $\mathscr{G}_A^\infty\backslash\mathscr{G}_A/\mathscr{G}_Q$ is finite.

Suppose first that G is a lattice nilpotent group. Then by Proposition 2.4, we have that there is an algebraic matric group $\mathfrak{G}$ such that $\mathscr{G}_A$ is isomorphic to $\mathfrak{G}_A$ by an isomorphism which takes $\mathscr{G}_A^\infty$ isomorphically onto $\mathfrak{G}_A^\infty$ and $\mathscr{G}_Q$ isomorphically onto $\mathfrak{G}_Q$. By Theorem A of §2, we have that the number of double cosets $\mathfrak{G}_A^\infty\backslash\mathfrak{G}_A/\mathfrak{G}_Q$ is finite so that the number of double cosets $\mathscr{G}_A^\infty\backslash\mathscr{G}_A/\mathscr{G}_Q$ is also finite.

Now suppose that G is any N-group. By a result of C. Moore [13], there is a lattice nilpotent group H, which contains G as a subgroup of finite index. By Lemma 1.8 we have that $Z_pG=Z_pH$ as subgroups of $Q_pG=Q_pH$ for all primes p except those in a finite set W. We have $QG=QH$ and $Q_pG=Q_pH$ as noted above for all primes p. For all primes p, except those in W, we have

$$\operatorname{stab}(Z_pG, \operatorname{Aut}(Q_pG)) = \operatorname{stab}(Z_pH, \operatorname{Aut}(Q_pG)) = \operatorname{stab}(Z_pH, \operatorname{Aut}(Q_pH)).$$

Suppose now that $\prod_{p\in V}(\alpha_p)$ is an element of $\mathscr{G}_A$, so that α_p is an element of stab $(Z_pG$, Aut $(Q_pG))$ for all primes p except for those in some finite set U. Then we have α_p in stab $(Z_pH$, Aut $(Q_pH))$ for all primes p except those in the finite set $U\cup W$, so that $\prod_{p\in V}(\alpha_p)$ is an element of $\mathscr{H}_A$. Thus we have $\mathscr{G}_A=\mathscr{H}_A$. Since $QG=QH$ we obviously have $\mathscr{G}_Q=\mathscr{H}_Q$. By Proposition 1.14 for each prime p in W, there are subgroups K_p of Aut (Q_pG) of finite index in each of stab $(Z_pG$, Aut $(Q_pG))$ and stab $(Z_pH$, Aut $(Q_pG))$. Thus

$$K = \prod_{p\in V\setminus W} \operatorname{stab}(Z_pG, \operatorname{Aut}(Q_pG)) \times \prod_{p\in W} K_p$$

is of finite index in each of $\mathscr{G}_A^\infty$ and $\mathscr{H}_A^\infty$. This implies that the number of double cosets

$$|\mathscr{G}_A^\infty\backslash\mathscr{G}_A/G_Q| \leqq |K\backslash\mathscr{G}_A/\mathscr{G}_Q| = |K\backslash\mathscr{H}_A/\mathscr{H}_Q|,$$

which is finite, since $|\mathscr{H}_A^\infty\backslash\mathscr{H}_A/\mathscr{H}_Q|$ is finite, and K is of finite index in $\mathscr{H}_A^\infty$.

The above computation together with Proposition 3.2 proves

THEOREM 3.3. *The number of isomorphism classes in the genus of an N-group is finite.*

This result, together with Theorem 3.1 of Borel, yields

THEOREM 3.4. *Let G be an N-group. Then there are only finitely many isomorphism classes of N-groups H for which $\mathscr{F}(G)=\mathscr{F}(H)$.*

PROPOSITION 3.5. *Let G and H be finitely generated nilpotent groups which have isomorphic finite quotients. Then the torsion subgroups τG and τH of G and H are isomorphic and the respective torsion-free factor groups $G^*=G/\tau G$ and $H^*=H/\tau H$ have isomorphic finite quotients.*

Proof. By Lemma 1.2, we have Z_pG isomorphic to Z_pH for each prime p. We will show that for each prime p, Z_pG^* is isomorphic to Z_pH^* and the p-Sylow subgroup of τG is isomorphic to the p-Sylow subgroup of τH. We have $\bigcap_{n=1}^\infty G^{p^n} =(\tau G)_{p'}$, the product of the q-Sylow subgroups of τG with $q\neq p$. This implies that the p-adic topology on $G_p=G/(\tau G)_{p'}$ will be Hausdorff and that $Z_pG_p=Z_pG$. We also have that $(\tau G)/(\tau G)_{p'}$ is the torsion subgroup of G_p and is normal. We have that $G_p/\tau(G_p)=G^*$, so that by Lemma 1.1, $Z_pG_p/Z_p\tau(G_p)$ is isomorphic to Z_pG^*. Since G^* is torsion-free, Z_pG^* is torsion-free, so that the torsion subgroup of Z_pG_p is $Z_p(\tau G_p)=\tau(G_p)=p$-Sylow subgroup of τG. Since Z_pG_p and Z_pH_p are isomorphic by hypothesis, their respective torsion subgroups must be isomorphic and their respective torsion-free factor groups must be isomorphic. That is, the p-Sylow subgroup of τG is isomorphic to the p-Sylow subgroup of τH and Z_pG^* is isomorphic to Z_pH^*. Since τG and τH are the direct product of their Sylow subgroups, τG and τH must be isomorphic and the proof is complete.

THEOREM 3.6. *Let G be a finitely generated nilpotent group. Then the finitely generated nilpotent groups H, for which $\mathscr{F}(G)=\mathscr{F}(H)$, are contained in finitely many isomorphism classes.*

Proof. By Theorem 3.4 and Proposition 3.5, it suffices to show that there can be only finitely many nonisomorphic groups with a given torsion subgroup τG and a given torsion-free quotient group G^*. Let $(x_1,\dots,x_n,r_1,\dots,r_m)$ be a presentation of G^*, and let $X_1,\dots,X_n$ be preimages of $x_1,\dots,x_n$ in an extension G of τG by G^*. G is then determined by the automorphisms α_i of τG given by conjugation by X_i and the elements $R_i=r_i(X_1,\dots,X_n)$ of τG. As there are only finitely many possible choices for each X_i, each α_i and each R_i, there can be only finitely many isomorphism classes of such extensions G. Q.E.D.

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